

The boundary of the Eulerian number triangle

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Abstract

The Eulerian triangle is a classical array of combinatorial numbers defined by a linear recursion. The associated boundary problem asks one to find all extreme nonnegative solutions to a dual recursion. Exploiting connections with random permutations and Markov chains we show that the boundary is discrete and explicitly identify its elements.

1 Introduction and main results

The *Eulerian triangle* (see, e.g., [8, Section 6.2]) is the infinite array of *Eulerian numbers*¹

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \quad (0 \leq k \leq n-1, \ n = 1, 2, \dots),$$

considered together with the defining them recursion

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = (k+1) \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle + (n-k) \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle \quad (1)$$

and the boundary conditions

$$\left\langle \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle = 1, \quad \text{and} \quad \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = 0 \quad \text{for } n < 0 \text{ or } k > n-1. \quad (2)$$

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¹Other commonly used notations are A_{nk} and E_{nk} . Some authors follow historical definition of Eulerian numbers as coefficients of Eulerian polynomials, hence index the numbers by k ranging from 1 to n .

The first six rows of the triangle ($n = 1, \dots, 6$) are

$$\begin{array}{ccccccc}
& & & & 1 & & \\
& & & & & 1 & \\
& & & 1 & & 4 & 1 \\
& & 1 & & 11 & & 11 & 1 \\
& 1 & & 26 & & 66 & & 26 & 1 \\
1 & & 57 & & 302 & & 302 & & 57 & 1
\end{array}$$

We are interested in *nonnegative* solutions

$$V = (V_{nk} : 0 \leq k \leq n-1, \quad n = 1, 2, \dots)$$

to the *dual* or *backward* recursion

$$V_{nk} = (k+1)V_{n+1,k} + (n-k)V_{n+1,k+1}, \quad (3)$$

subject to the only normalization condition $V_{10} = 1$. In contrast to (1), the dual recursion has multiple solutions which comprise a convex set \mathcal{V} . We denote by $\text{ex}(\mathcal{V})$ the set of extreme solutions and call it the *boundary*.

One principal result of this paper gives a natural parametrization of the boundary:

Theorem 1. *Every extreme solution $W = (W_{nk}) \in \text{ex}(\mathcal{V})$ is uniquely determined by the parameter $\theta := W_{20}$, which assumes values in the following subset of the unit interval*

$$\Theta = \left\{ \frac{1}{2} \cdot \frac{\varkappa}{\varkappa+1}, \quad \varkappa = 0, 1, \dots \right\} \cup \left\{ \frac{1}{2} \right\} \cup \left\{ \frac{1}{2} \cdot \frac{\varkappa+2}{\varkappa+1}, \quad \varkappa = 0, 1, \dots \right\}.$$

The correspondence between $\text{ex}(\mathcal{V})$ and Θ is a homeomorphism.

Thus, the parameter set Θ is composed of two sequences and their sole accumulation point $\frac{1}{2}$. An obvious symmetry of Θ about $\frac{1}{2}$ corresponds to the symmetry of (1) under the substitution $(n, k) \rightarrow (n, n-1-k)$.

Theorem 1 distinguishes the Eulerian triangle from other classical number triangles, whose boundaries can be also identified with subsets of the unit interval. It is well known that the boundary of the Pascal triangle is $[0, 1]$; a result equivalent to both de Finetti's theorem on exchangeable trials and Hausdorff's characterization of moment sequences [1, 3, 12, 23]. The boundary

of the q -Pascal triangle of q -binomial coefficients is $\{0\} \cup \{\dots, q^{-3}, q^{-2}, q^{-1}\}$ for $q > 1$, see [11, 17]. For a parametric family of the generalized Stirling triangles considered in [7] the boundary was shown discrete (as for the q -Pascal triangle) for some values of the parameter and coinciding with $[0, 1]$ for other. For instance, the boundary is discrete for the triangle of Stirling numbers of the second kind, and it is continuous for the triangle of signless Stirling numbers of the first kind. Some sporadic earlier results on a wider family of generalized Stirling triangles are found in [10].

The extreme solution corresponding to a particular value of the parameter $\theta \in \Theta$ will be denoted $W(\theta) = (W_{nk}(\theta))$. Our second principal result gives explicit formulas for these solutions:

Theorem 2. *For $\theta = \frac{\varkappa+2}{2(\varkappa+1)} > \frac{1}{2}$ with $\varkappa = 0, 1, 2, \dots$ we have*

$$W_{nk}(\theta) = \binom{n + \varkappa - k}{n} \Big/ (\varkappa + 1)^n \quad (n = 1, 2, \dots; 0 \leq k \leq n - 1), \quad (4)$$

whereas for $\theta = \frac{\varkappa}{2(\varkappa+1)} < \frac{1}{2}$ with $\varkappa = 0, 1, 2, \dots$ we have

$$W_{nk}(\theta) = \binom{\varkappa + k + 1}{n} \Big/ (\varkappa + 1)^n \quad (n = 1, 2, \dots; 0 \leq k \leq n - 1). \quad (5)$$

Finally, the solution corresponding to $\theta = \frac{1}{2}$ is

$$W_{nk}(\tfrac{1}{2}) = \frac{1}{n!} \quad (n = 1, 2, \dots; 0 \leq k \leq n - 1). \quad (6)$$

It is seen that for $\theta > \frac{1}{2}$, $W_{nk}(\theta)$ is 0 for $k > \varkappa$, and for $\theta < \frac{1}{2}$ it is 0 for $k < n - 1 - \varkappa$. Notice also the symmetry

$$W_{nk}(\theta) = W_{n, n-1-k}(1 - \theta).$$

All three formulas of Theorem 2 can also be written in a unified way

$$W_{nk}(\theta) = \frac{1}{n!} \prod_{i=-k}^{-k+n-1} (1 + \theta' i), \quad (7)$$

where $\theta' := 2\theta - 1$ ranges over the set

$$\Theta' = \{-1, -\tfrac{1}{2}, -\tfrac{1}{3}, -\tfrac{1}{4}, \dots, 0, \dots, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, 1\}.$$

Theorems 1 and 2 are our main results. They imply a simple description of the whole set of nonnegative normalized solutions to (3).

Corollary 3. *Each solution $V \in \mathcal{V}$ can be uniquely represented as a convex combination*

$$V = \sum_{\theta \in \Theta} p(\theta) W(\theta),$$

with p a probability distribution on Θ .

The rest of the paper is organized as follows. In Section 2 we encode the Eulerian triangle into a graded graph \mathcal{E} , which allows to embed our boundary problem in the general framework developed in [10, 11, 13, 24]. In particular, we recall a construction of the Martin boundary, which is our major technical tool. In Section 3 we explain connections to random permutations and random total orders on $\mathbb{N} = \{1, 2, \dots\}$ with a sufficiency property. In Section 4 the extreme solutions are related to random sorting algorithms. The main results are proved in Sections 5 and 6. Some connections and extensions are discussed in the last section.

Boundary problems of combinatorial nature arise within a variety of mathematical disciplines like probability, numerical analysis, algebra and representation theory. Our own view on the subject has two major sources: the asymptotic representation theory of symmetric groups initiated by Kerov and Vershik [24] and Kingman's theory of exchangeable partitions summarized in lectures by Pitman [22].

2 The graph \mathcal{E} and the Martin boundary

The graph \mathcal{E} is an infinite directed graded graph with the vertex set

$$\{(n, k); n = 1, 2, \dots, 0 \leq k \leq n - 1\}$$

and multiple edges. The n th level of \mathcal{E} , denoted \mathcal{E}_n , is the set of n vertices $(n, 0), \dots, (n, n - 1)$. The level \mathcal{E}_1 has a single vertex $(1, 0)$, which is the root of \mathcal{E} . For generic vertex (n, k) the outgoing edges link (n, k) to two vertices $(n + 1, k)$ and $(n + 1, k + 1)$ by $k + 1$ and $n - k$ directed edges, respectively. We say that two vertices connected by an edge are *adjacent* with each other. A path in \mathcal{E} (finite or infinite) starting at (n, k_n) is a sequence of edges linking adjacent vertices $(n, k_n), (n + 1, k_{n+1}), \dots$ on consecutive levels. A

path starting at the root $(1, 0)$ is called *standard*. The edge multiplicities in \mathcal{E} are selected to match with the coefficients in the recursion (1), from which it is clear that the Eulerian number $\langle n \rangle_k$ is the *dimension*² of the vertex (n, k) , meaning the number of standard paths in \mathcal{E} which terminate at (n, k) .

The interpretation of dual recursion (3) requires some concepts of probability theory. Let us consider \mathcal{E} as a state space of some Markov process whose time parameter n runs in the *reverse* direction $\cdots \rightarrow n \rightarrow n-1 \rightarrow \cdots \rightarrow 1$, a possible state at time n is a vertex in \mathcal{E}_n , and the transition probabilities are given by

$$\text{Prob}\{(n, k) \rightarrow (n-1, k)\} = (k+1) \frac{\langle n-1 \rangle_k}{\langle n \rangle_k} \quad (8)$$

$$\text{Prob}\{(n, k) \rightarrow (n-1, k-1)\} = (n-k) \frac{\langle n-1 \rangle_{k-1}}{\langle n \rangle_k}. \quad (9)$$

The basic relations (1) and (2) translate as the rule of total probability

$$\text{Prob}\{(n, k) \rightarrow (n-1, k)\} + \text{Prob}\{(n, k) \rightarrow (n-1, k-1)\} = 1, \quad (10)$$

and imply that at consecutive times the process must reside in adjacent vertices of \mathcal{E} .

Now let $V = (V_{nk}) \in \mathcal{V}$. Setting

$$\tilde{V}_{nk} = \langle n \rangle_k V_{nk}, \quad (11)$$

the recursion (3) can be rewritten as

$$\begin{aligned} \tilde{V}_{nk} &= \text{Prob}\{(n+1, k) \rightarrow (n, k)\} \tilde{V}_{n+1, k} \\ &\quad + \text{Prob}\{(n+1, k+1) \rightarrow (n, k)\} \tilde{V}_{n+1, k+1} \end{aligned} \quad (12)$$

Lemma 4. *We have*

$$\sum_{k=0}^{n-1} \tilde{V}_{nk} = 1, \quad n = 1, 2, \dots \quad (13)$$

²The dimensions alone do not determine \mathcal{E} unambiguously, since the Eulerian numbers satisfy many other recursions different from (1). By our definition of the Eulerian triangle we were careful to say that the array was considered together with the recursion (1).

Proof. Indeed, from (10) and (12), the quantity $\sum_{k=0}^{n-1} \tilde{V}_{nk}$ does not depend on n . Since it equals 1 for $n = 1$ due to the normalization condition, the same holds for all n . \square

Thus, the vector $(\tilde{V}_{n0}, \dots, \tilde{V}_{n,n-1})$ is a probability distribution on \mathcal{E}_n for each n , and this family of distributions is consistent with the transition probabilities (8) and (9). It follows that V determines the law of a Markov chain by the virtue of (11). The boundary problem acquires therefore the following meaning:

- describe all possible probability laws for a Markov chain on \mathcal{E} , whose transition probabilities are given by (8) and (9).

If we required (3) to only hold for n restricted to some finite range $1 \leq n \leq N$, the analogous boundary problem were rather simple. For, each truncated solution

$$(V_{nk}; \quad 0 \leq k \leq n-1, \quad n = 1, \dots, N)$$

is uniquely determined by the last row $(V_{N0}, \dots, V_{N,N-1})$. Equivalently, the corresponding Markov chain with time parameter n ranging from N to 1 is determined by the initial distribution $(\tilde{V}_{N0}, \dots, \tilde{V}_{N,N-1})$, which can be selected arbitrarily within the set of all probability distributions on \mathcal{E}_N . The set $\mathcal{V}^{(N)}$ of (nonnegative, normalized) solutions to such a truncated recursion is therefore the convex hull of the arrays $V^{N\kappa}$ ($0 \leq \kappa \leq N-1$) which have the N th row

$$V_{Nk}^{N\kappa} = \delta_{\kappa k} \bigg/ \left\langle \begin{matrix} N \\ \kappa \end{matrix} \right\rangle \quad (0 \leq k \leq N-1),$$

where $\delta_{\kappa k}$ is the Kronecker symbol. The set $\mathcal{V}^{(N)}$ is a $(N-1)$ -dimensional simplex with extreme elements

$$\text{ex}(\mathcal{V}^{(N)}) = \{V^{N\kappa} : \quad \kappa = 0, \dots, N-1\}.$$

The probability law $\tilde{V}^{N\kappa}$ corresponding to $V^{N\kappa}$ rules a Markov chain which starts in state $(N, \kappa) \in \mathcal{E}_N$ at time N , hence the boundary of the N -truncated triangle can be identified with \mathcal{E}_N .

For the infinite recursion the problem is much more complicated because there is no obvious analogue of the “last row” which would provide an initial

condition for (3). A common recipe to obtain all solutions is the following. Extend each $V^{N\kappa}$ to a function on the whole set of vertices of \mathcal{E} by setting $V_{nk}^{N\kappa} = 0$ for $n > N$. Define the *Martin boundary*³ of \mathcal{E} as

$$\partial\mathcal{V} := \overline{\left(\bigcup_{N,\kappa} \mathcal{V}^{N\kappa}\right)} \setminus \left(\bigcup_{N,\kappa} \mathcal{V}^{N\kappa}\right),$$

where the bar means the closure in the topology of pointwise convergence of functions on the set of vertices. Plainly, $\partial\mathcal{V}$ is the set of solutions which may be obtained from truncated solutions $V^{N\kappa}$ by fixing some limiting regime for $\kappa = \kappa(N)$, as $N \rightarrow \infty$, to secure convergence of $V_{nk}^{N\kappa}$ for each (n, k) . Obviously, each such limit is indeed a solution to (3), hence $\partial\mathcal{V}$ is a subset of \mathcal{V} .

By some well known general theory (see [11, Ch. 1, §1] and also [1, 4, 9, 12]) the Martin boundary $\partial\mathcal{V}$ contains the boundary $\text{ex}(\mathcal{V})$ (for this reason $\partial\mathcal{V}$ is sometimes called the *maximal* boundary). In our proof of Theorems 1 and 2 we shall determine the Martin boundary $\partial\mathcal{V}$ and then check that all its elements are actually extreme solutions.

The coincidence of boundaries is not specific for \mathcal{E} , rather it holds for other number triangles and more sophisticated graded graphs [13]. A common reason for this phenomenon is some law of large numbers, like the law of large numbers for exchangeable 0–1 random variables in the case of Pascal triangle. On the other hand, there are simple examples of graded graphs for which the Martin boundary is strictly larger than the extreme boundary [7].

3 D–arrangements

Let $[n] = \{1, \dots, n\}$ and let $\text{Perm}(n)$ be the set of permutations of $[n]$. We write permutations $\pi \in \text{Perm}(n)$ in the conventional one–row notation $\pi = \pi(1) \dots \pi(n)$ (and ignore the group structure on $\text{Perm}(n)$). A position $j \in [n-1]$ is said to be a *descent* of π if $\pi(j) > \pi(j+1)$. By $D(\pi)$ we denote the total number of descents of π . For instance, $\pi = 7356241 \in \text{Perm}(7)$ has descents at positions $j = 1, 4, 6$, hence $D(7356241) = 3$.

According to a well–known combinatorial interpretation, the Eulerian

³The definition corresponds to the *entrance* boundary in [9].

numbers count permutations with a given number of descents:

$$\text{Card}\{\pi \in \text{Perm}(n) : D(\pi) = k\} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle, \quad (14)$$

as is easily shown by checking that the counts satisfy the recursion (1) (or see [8, Section 6.2]). We establish next a more delicate connection.

Observe that removing n from a permutation of $[n]$ yields a projection $p_n : \text{Perm}(n) \rightarrow \text{Perm}(n-1)$. For instance, $3412 \in \text{Perm}(4)$ is projected to $312 \in \text{Perm}(3)$. Clearly, the preimage of any permutation $\pi \in \text{Perm}(n-1)$ by p_n consists of exactly n elements.

Lemma 5. *There exists a bijection b_n between $\text{Perm}(n)$ and standard paths in the graph \mathcal{E} of length n with the following property: the path $b_n(\pi_n)$ corresponding to $\pi_n \in \text{Perm}(n)$ passes through the vertices $(m, D(\pi_m)) \in \mathcal{E}_m$ ($m = 1, \dots, n$), where $\pi_{n-1} = p_n(\pi_n), \dots, \pi_1 = p_2(\pi_2)$ are the iterated projections of π_n .*

Proof. Choose $\pi_n \in \text{Perm}(n)$ and let $k = D(\pi_n)$. It is readily checked that the preimage $p_{n+1}^{-1}(\pi_n) \subset \text{Perm}(n+1)$ consists of $k+1$ permutations with k descents and of $n-k$ permutations with $k+1$ descents. Observe that $k+1$ is the number of edges linking (n, k) to $(n+1, k)$ while $n-k$ is the number of edges linking (n, k) to $(n+1, k+1)$. It follows that if the desired bijection exists for some n then it can be further extended to a bijection for $n+1$. The assertion follows by induction. \square

The bijections b_n are in no way canonical, because we do not distinguish among the edges linking adjacent vertices in \mathcal{E} . Still, the way we introduced b_n 's takes care of *consistency* for all n . Indeed, let t_n be the operation of cutting off the last link in a standard path in \mathcal{E} of length n . Thus t_n projects standard paths of length n onto standard paths of length $n-1$. The consistency of b_n 's amounts to the commutation relation $t_n \circ b_n = b_{n-1} \circ p_n$, which holds for all $n \geq 2$.

Let $\mathfrak{A} = \varprojlim \text{Perm}(n)$ be the inverse limit ⁴ of the finite permutation spaces $\text{Perm}(n)$ with respect to p_n 's. Elements of \mathfrak{A} are infinite sequences (π_n) of *consistent* permutations $\pi_n \in \text{Perm}(n)$ ($n = 1, 2, \dots$), meaning that, for each $n \geq 2$, $p_n(\pi_n) = \pi_{n-1}$. In extension of Lemma 5 we have the following corollary.

⁴Another inverse limit, the space of virtual permutations, appears in [14].

Corollary 6. *The consistent sequence of bijections (b_n) defines a bijection between \mathfrak{A} and the set of infinite standard paths in \mathcal{E} . The bijection has the property that the path corresponding to $(\pi_n) \in \mathfrak{A}$ passes through the vertices $(n, D(\pi_n))$, $n = 1, 2, \dots$*

With each permutation $\pi \in \text{Perm}(n)$ we associate a total order \triangleleft on the set $[n]$, in which $\pi(1) \triangleleft \dots \triangleleft \pi(n)$. Likewise, every element $(\pi_n) \in \mathfrak{A}$ determines a total order on the set $\mathbb{N} = \{1, 2, \dots\}$ such that, for each n , the total order restricted to the subset $[n] \subset \mathbb{N}$ is the one given by π_n . Conversely, any total order on \mathbb{N} can be obtained in this way, from some element of \mathfrak{A} . For this reason, we call the elements of \mathfrak{A} *arrangements* and identify them with the total orders on \mathbb{N} . Two obvious examples of arrangements are the standard order $1 \triangleleft 2 \triangleleft 3 \triangleleft \dots$ and the inverse order $\dots \triangleleft 3 \triangleleft 2 \triangleleft 1$; the corresponding paths in \mathcal{E} go along the left side of the Euler triangle and along its right side, respectively.

As a projective limit of finite sets, \mathfrak{A} is a compact topological space. Given a probability measure P on \mathfrak{A} we can speak of a *random arrangement* $\Pi = (\Pi_n)$, where $\Pi_n \in \text{Perm}(n)$ are consistent random permutations, such that the law P_n of Π_n is the pushforward of P by the canonical projection $\mathfrak{A} \rightarrow \text{Perm}(n)$. Conversely, by Kolmogorov's measure extension theorem, each sequence of distributions (P_n) determines a unique random arrangement, provided the sequence is consistent with respect to all projections p_n .

The random arrangements relevant to our discussion have one special property of sufficiency.

Definition 7. We say that a random arrangement $\Pi = (\Pi_n)$ is a *D-arrangement* if for every $n = 1, 2, \dots$ and $\pi_n \in \text{Perm}(n)$ the probability of the event $\Pi_n = \pi_n$ depends on the couple $(n, D(\pi_n))$ only.

That is to say, for a D-arrangement $\Pi = (\Pi_n)$, the number of descents is a *sufficient statistic*: conditionally given $D(\Pi_n) = k$ the distribution of Π_n is uniform on the set of permutations of $[n]$ with k descents, for each n and k .

Two trivial examples of D-arrangements are the nonrandom arrangements given by the standard order and the inverse order. The corresponding measures on \mathfrak{A} are the Dirac masses at points $(\pi_n) = (1 \dots n) \in \mathfrak{A}$ and $(\pi_n) = (n \dots 1) \in \mathfrak{A}$, respectively. Notice that these two are the only Dirac measures on \mathfrak{A} corresponding to D-arrangements.

More substantial example is the random arrangement Π for which every Π_n has uniform distribution on $\text{Perm}(n)$. This is the only *exchangeable* random arrangement, whose probability law is invariant under arbitrary permutations of the set \mathbb{N} .

Now, Corollary 6 implies:

Lemma 8. *The formula*

$$P_n(\pi_n) = V_{n,D(\pi_n)}, \quad n = 1, 2, \dots, \quad \pi_n \in \text{Perm}(n)$$

defines an affine isomorphism $V = (V_{nk}) \leftrightarrow P = (P_n)$ between \mathcal{V} and the set of probability laws for D -arrangements.

Equivalently, in terms of quantities \tilde{V}_{nk} and random paths in \mathcal{E} corresponding to D -arrangements, \tilde{V}_{nk} is the probability that a random infinite path (with distribution P) will pass through the vertex (n, k) . In the sequel we will not distinguish between solutions to (3) and random D -arrangements.

4 Bucket sorting

Here we use the correspondence of Lemma 8 for constructing a family of solutions $V \in \mathcal{V}$. The following algorithm, called *bucket sorting*, exploits a multinomial distribution and is a simplest of the algorithms of this kind, widely known in computer science [16] and dynamical systems [2, 15].

Fix $\varkappa \in \{0, 1, 2, \dots\}$ and imagine $\varkappa + 1$ buckets arranged in some order. Suppose each of the numbers $1, 2, \dots \in \mathbb{N}$ is sent to one of the buckets with equal probabilities $(\varkappa + 1)^{-1}, \dots, (\varkappa + 1)^{-1}$, independently of the other numbers. For each n this yields a random allocation of integers $1, \dots, n$ in the buckets. Arranging the integers within each bucket in increasing order and putting the resulting sequences together (in the order of the buckets), the allocation of n integers is transformed into a random permutation Π_n^\varkappa of $[n]$. By the construction, Π_n^\varkappa has at most \varkappa descents.

Lemma 9. *The infinite sequence $\Pi^\varkappa = (\Pi_n^\varkappa)$ produced by the bucket sorting is a D -arrangement. The corresponding array $W^\varkappa = (W_{nk}^\varkappa) \in \mathcal{V}$ is given by formula*

$$W_{nk}^\varkappa = \binom{n + \varkappa - k}{n} / (\varkappa + 1)^n. \quad (15)$$

Proof. By the very construction, the random permutations Π_n^\varkappa are consistent with respect to the projections p_n , hence Π^\varkappa is a random arrangement. Given $\pi_n \in \text{Perm}(n)$, let us compute the probability of the event $\Pi_n^\varkappa = \pi_n$. The total number of possible allocations of $1, \dots, n$ into buckets equals $(\varkappa + 1)^n$, and all of them are equally likely. Thus, it suffices to compute the number of the allocations resulting in π_n . Any such allocation is determined by a partition of the sequence $\pi_n = \pi_n(1) \dots \pi_n(n)$ into $\varkappa + 1$ consecutive fragments (some of which can be empty), and any such partition can be encoded by placing \varkappa vertical bars separating the fragments. Observe that for each descent $j \in [n - 1]$ of the permutation π_n at least one bar has to be placed between $\pi_n(j)$ and $\pi_n(j + 1)$. For $k = D(\pi_n)$ we see that k positions of bars are fixed by the descents, so that the allocation is actually determined by the remaining $\varkappa - k$ bars, which can be placed arbitrarily. Since the bars are indistinguishable, the number of possibilities equals $\binom{n + \varkappa - k}{n}$. Thus, the probability of π_n is given by the right-hand side of (15). Since this expression depends only on $k = D(\pi_n)$, we conclude that Π^\varkappa is a D-arrangement and (15) is the corresponding element of \mathcal{V} . \square

Remark 10. The fact that formula (15) determines a solution to (3) amounts to a binomial identity which is easy to check directly:

$$(\varkappa + 1) \binom{n + \varkappa - k}{n} = (k + 1) \binom{n + \varkappa + 1 - k}{n + 1} + (n - k) \binom{n + \varkappa - k}{n + 1},$$

while the total probability rule (13) becomes

$$(\varkappa + 1)^n = \sum_{k=0}^{n-1} \langle n \rangle_k \binom{n + \varkappa - k}{n},$$

which is equivalent to Worpitzky's identity [8, (6.37)].

Recall that in Section 2 we introduced arrays $V^{N\varkappa} = (V_{nk}^{N\varkappa}) \in \mathcal{V}^{(N)}$ solving the “ N -truncated” version of recursion (3).

Lemma 11. Fix $\varkappa \in \{0, 1, \dots\}$, let $\Pi^\varkappa = (\Pi_n^\varkappa)$ be the D-arrangement resulting from the bucket sorting, and let $W^\varkappa = (W_{nk}^\varkappa) \in \mathcal{V}$ stand for the corresponding array. Then $V^{N\varkappa}$ converge to W^\varkappa , that is

$$\lim_{N \rightarrow \infty} V_{nk}^{N\varkappa} = W_{nk}^\varkappa \quad (n = 1, 2, \dots; 0 \leq k \leq n - 1).$$

Proof. It is more convenient to deal with quantities

$$\widetilde{V}_{nk}^{N\kappa} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle V_{nk}^{N\kappa}, \quad \widetilde{W}_{nk}^{\kappa} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle W_{nk}^{\kappa}.$$

For fixed n , the vectors

$$(\widetilde{V}_{n0}^{N\kappa}, \dots, \widetilde{V}_{n,n-1}^{N\kappa}) \quad \text{and} \quad (\widetilde{W}_{n0}^{\kappa}, \dots, \widetilde{W}_{n,n-1}^{\kappa})$$

are the distributions at time n of the N -step Markov chain (introduced in Section 2) whose initial distribution at time N is

$$(\widetilde{V}_{N0}^{N\kappa}, \dots, \widetilde{V}_{N,N-1}^{N\kappa}) \quad \text{or} \quad (\widetilde{W}_{N0}^{\kappa}, \dots, \widetilde{W}_{N,N-1}^{\kappa}), \quad (16)$$

respectively.

Recall that both $\widetilde{V}_{Nk}^{N\kappa}$ and $\widetilde{W}_{Nk}^{\kappa}$ vanish for $k > \kappa$ and, moreover,

$$\widetilde{V}_{Nk}^{N\kappa} = \delta_{\kappa k}. \quad (17)$$

We claim that it suffices to prove the limit relation

$$\lim_{N \rightarrow \infty} \widetilde{W}_{N\kappa}^{\kappa} = 1. \quad (18)$$

Indeed, since (16) are probability distributions, (17) and (18) imply that the total variance distance between them goes to 0 as $N \rightarrow \infty$, which implies the assertion of the lemma.

To prove (18) we turn to Π^{κ} and observe that $\widetilde{W}_{N\kappa}^{\kappa}$ is just the probability for the random permutation Π_N^{κ} to have the maximal possible number of descents κ . In terms of the random allocation of the numbers $1, \dots, N$, this means that all buckets are nonempty and the largest number in each bucket (except the last bucket) is larger than the smallest number in the next bucket. If this were not the case, all the numbers in one of the buckets were smaller than those in the next bucket. Elementary estimates which we postpone to the proof of Lemma 13 show that the probability of such an event tends to 0 as $N \rightarrow \infty$, which yields (18). \square

Remark 12. By virtue of the explicit formula (15), the limit relation (18) is equivalent to an asymptotic relation for the Eulerian numbers:

$$\lim_{N \rightarrow \infty} \left\langle \begin{matrix} N \\ \kappa \end{matrix} \right\rangle \sim (\kappa + 1)^N \quad \text{for fixed } \kappa = 0, 1, \dots$$

This relation can be readily checked directly. For instance, it follows from formula (22) below.

Lemma 11 shows that the family $\{W^\varkappa\}$ is contained in the Martin boundary. Actually, a stronger claim holds: all W^\varkappa 's are extreme. We show this in Lemma 14 below. But first we will prove the law of large numbers for Π^\varkappa .

Lemma 13. *Fix $\varkappa = \{0, 1, \dots\}$. We have*

$$\lim_{n \rightarrow \infty} D(\Pi_n^\varkappa) = \varkappa \quad \text{with probability 1.}$$

Moreover, this property of Π^\varkappa is characteristic.

Proof. Suppose there are just two buckets, $\varkappa = 2$. Then $D(\Pi_n^2) < 2$ means that for some $m \in [n]$ the integers $1, \dots, m$ fall in the first bucket, and $m+1, \dots, n$ in the second, which is an event of probability $(n+1)/2^n$. Since the series of these probabilities converges, the Borel–Cantelli lemma yields the claim. The general case $\varkappa > 2$ is reduced to the estimate in the case $\varkappa = 2$ by focussing on two consecutive buckets and using elementary large deviation bounds for Bernoulli trials to show that the chance for less than, say, n/\varkappa integers in both buckets goes to 0 exponentially fast with n . The uniqueness follows as in Lemma 11. \square

Lemma 14. *Elements $W^\varkappa \in \mathcal{V}$ resulting from the bucket sorting are extreme.*

Proof. If W^\varkappa is a mixture of some $V^1, V^2 \in \mathcal{V}$ then by the first assertion of Lemma 13 the arrangements corresponding to V_1 and V_2 must satisfy the same law of large numbers as W^\varkappa . But then by the second assertion of the lemma $V_1 = V_2 = W^\varkappa$. Hence W^\varkappa is extreme. \square

5 Proofs of the main results

We start with reducing the set of parameters needed to determine a generic solution $V \in \mathcal{V}$.

Lemma 15. *The sequence (V_{n0}) uniquely determines $V \in \mathcal{V}$.*

Proof. Writing (3) as

$$V_{n+1,k+1} = \frac{1}{n-k} V_{nk} - \frac{k+1}{n-k} V_{n+1,k} \quad (0 \leq k \leq n-1),$$

we see that for each $k \in \{0, 1, \dots\}$ the sequence $(V_{n,k+1} : n = k+2, k+3, \dots)$ is uniquely determined by the sequence $(V_{nk} : n = k+1, k+2, \dots)$. Induction in k ends the proof. \square

Consider the truncated arrays $V^{N\kappa} = (V_{nk}^{N\kappa}) \in \mathcal{V}^{(N)}$ with parameters N and κ , $0 \leq \kappa \leq N - 1$, introduced in Section 4. As in the proof of Lemma 11 we introduce the modified array $\tilde{V}^{N\kappa} = (\tilde{V}_{nk}^{N\kappa})$ with $\tilde{V}_{nk}^{N\kappa} = \langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle V_{nk}^{N\kappa}$, and we recall that the n th row of $\tilde{V}^{N\kappa}$ is the distribution at time n of the Markov chain started at time N from the vertex (N, κ) . Since $V_{n0}^{N\kappa} = \tilde{V}_{n0}^{N\kappa}$, the quantity $V_{n0}^{N\kappa}$ equals the probability of the event that the Markov chain will pass through vertex $(n, 0)$.

Furthermore, there is a monotonicity property analogous to that of generalized Stirling triangles in [7].

Lemma 16. *For fixed $n \in \{1, \dots, N\}$, the coordinate $V_{n0}^{N\kappa}$ does not increase as κ varies from 0 to $N - 1$.*

Proof. We employ a simple coupling argument. Given two numbers $\kappa < \kappa'$, consider two Markov chains which start from vertices (N, κ) and (N, κ') , respectively. We settle both chains on a common probability space assuming that the jumps are independent as long as the trajectory of the first chain does not intersect the trajectory of the second chain, but once the trajectories meet, they merge. The merge does not affect the marginal law of each of the chains, since both are directed by the same transition probabilities. The key property of the coupling is that each trajectory of the first chain remains on the left of the trajectory of the second chain, before the trajectories merge. Observe now that after reaching the left side of the Euler triangle, a trajectory can only process along this side. Consequently, if a trajectory of the second chain passes through $(n, 0)$ then the trajectory of the first chain reaches the left side of the triangle at some time $n' \geq n$, hence passes through $(n, 0)$, too. Thus, the chance for the first chain to pass through $(n, 0)$ is not less than that for the second chain. This proves the desired inequality $V_{n0}^{N\kappa} \geq V_{n0}^{N\kappa'}$. \square

We proceed with the proof of Theorems 1 and 2. Our strategy is to determine first the Martin boundary by directly identifying all solutions W that appear as limits of arbitrary sequences of the form $V^{N, \kappa(N)}$.

Assume first that $\kappa(N) = \kappa$ with some fixed κ , for all N large enough. Then, by Lemma 11, the sequence $V^{N, \kappa(N)}$ converges to the array W^κ given by formula (15) which we also display here for reader's convenience:

$$W_{nk}^\kappa = \binom{n + \kappa - k}{n} / (\kappa + 1)^n. \quad (19)$$

Next, assume that $\varkappa(N) = N - 1 - \varkappa$, where \varkappa is fixed. Observe that this limit regime is reduced to the preceding one by application of the symmetry $(n, k) \rightarrow (n, n - 1 - k)$ of the Euler triangle \mathcal{E} . Therefore, in this case the sequence converges to the array \widehat{W}^\varkappa with components

$$\widehat{W}_{nk}^\varkappa = \binom{\varkappa + k + 1}{n} / (\varkappa + 1)^n. \quad (20)$$

The random D-arrangement corresponding to \widehat{W}^\varkappa can be produced by the obvious analogue of bucket sorting in which integers within each bucket are arranged in *decreasing* order.

Further on, from (19) and (20) it is readily seen that there exists the limit

$$\lim_{\varkappa \rightarrow \infty} W^\varkappa = \lim_{\varkappa \rightarrow \infty} \widehat{W}^\varkappa = W^\infty$$

with components

$$W_{nk}^\infty = \frac{1}{n!}.$$

Clearly, $W^\infty \in \mathcal{V}$.

Now we claim that if both $\varkappa(N)$ and $N - 1 - \varkappa(N)$ go to infinity then the sequence $V^{N, \varkappa(N)}$ converges to W^∞ . To that end, observe that a general bound

$$0 \leq V_{nk} \leq \left\langle \frac{n}{k} \right\rangle^{-1}, \quad (21)$$

which follows from Lemma 4, holds for all $V \in \mathcal{V}$ and implies that \mathcal{V} is compact in the product topology. By the compactness, passing if necessary to a subsequence of $(V^{N, \varkappa(N)})$ we can always achieve convergence to some W , hence it is enough to show that $W = W^\infty$, and by Lemma 15, this is further reduced to showing that $W_{n0} = W_{n0}^\infty = \frac{1}{n!}$. For any fixed \varkappa we have

$$\varkappa \leq \varkappa(N) \leq N - 1 - \varkappa$$

for large N . Applying Lemma 16 we obtain the bound

$$W_{n0}^\varkappa \geq W_{n0} \geq \widehat{W}_{n0}^\varkappa \quad (n = 1, 2, \dots).$$

Now, sending \varkappa to infinity we conclude that $W_{n0} = \frac{1}{n!}$, as wanted.

We have shown that the Martin boundary consists of the elements

$$W^\varkappa \quad (\varkappa = 0, 1, 2, \dots), \quad \widehat{W}^\varkappa \quad (\varkappa = 0, 1, 2, \dots), \quad \text{and} \quad W^\infty.$$

Comparison with formulas of Theorem 1 shows that these are exactly the arrays $W(\theta)$ with $\theta > \frac{1}{2}$, $\theta < \frac{1}{2}$, and $\theta = \frac{1}{2}$, respectively. A remarkable fact emerges: the single entry $(2, 0)$ distinguishes all these arrays.

By Lemma 14, the arrays W^\varkappa are extreme. By symmetry, the arrays \widehat{W}^\varkappa are extreme, too. To finish the proof of the theorems it remains to check that $W^\infty = W(\frac{1}{2})$ is extreme. We postpone this to the next section.

The Corollary 3 follows from general results. Since the space \mathcal{V} is compact, metrizable and separable, the well-known Choquet theorem [20, §3] implies that each solution $V \in \mathcal{V}$ may be represented as a convex mixture of the extreme solutions $W \in \text{ex}(\mathcal{V})$. A simple general argument shows that \mathcal{V} is a Choquet simplex (that is, the cone generated by \mathcal{V} is a lattice cone), see, e.g., [18, Lemma 9.3]. The uniqueness of representation now follows from another Choquet's theorem, see [20, §9].

6 End of proof: extremality of the exchangeable arrangement

In this section P denotes the probability measure on \mathfrak{A} corresponding to the array $W^\infty = W(\frac{1}{2}) \in \mathcal{V}$. The characteristic property of P is that, for each n , the image P_n under the natural projection $\mathfrak{A} \rightarrow \text{Perm}(n)$ is the uniform measure assigning to all permutations $\pi_n \in \text{Perm}(n)$ equal weights $\frac{1}{n!}$. Let Π be the random arrangement with law P . This is the exchangeable random arrangement, invariant under the natural action on the space \mathfrak{A} of permutations of \mathbb{N} .

The following useful construction of Π is found in [1]. Let X_1, X_2, \dots be independent random variables, with uniform distribution on the unit interval $[0, 1]$. The X_j 's are pairwise distinct with probability one. Define a random total order on \mathbb{N} by the rule $i \triangleleft j$ if $X_i < X_j$. Clearly, for each n , the resulting random permutation Π_n of $[n]$ depends only of X_1, \dots, X_n and, by exchangeability of X_j 's, Π_n is uniformly distributed on $\text{Perm}(n)$.

Known moments of $D(\Pi_n)$ follow easily from this realization.

Lemma 17. *Let Π_n be the uniform random permutation of $[n]$. The random variable $D(\Pi_n)$ has mean $(n-1)/2$ and variance $(n-1)/12$.*

Proof. Clearly, $D(\Pi_n)$ equals the number of descents in the random sequence X_1, \dots, X_n , that is, the number of indices $i \in [n-1]$ such that $X_i > X_{i+1}$.

Therefore, denoting by χ_i the indicator of the event $X_i > X_{i+1}$ we have

$$D(\Pi_n) = \sum_{i=1}^{n-1} \chi_i.$$

The result easily follows from the relations

$$\mathbb{E}(\chi_i) = \mathbb{E}(\chi_i^2) = \frac{1}{2}, \quad \mathbb{E}(\chi_i \chi_{i+1}) = \frac{1}{6}, \quad \mathbb{E}(\chi_i \chi_j) = 0 \quad (|i - j| \geq 2).$$

□

Since both the mean and the variance exhibit a linear growth, standard application of Chebyshev's inequality gives:

Corollary 18. *Under the uniform distribution, $D(\Pi_n)/(n-1) \rightarrow \frac{1}{2}$ in probability.*

We have now all tools to finish the argument of Section 5 by showing that W^∞ is extreme. Assume the contrary, then the boundary $\text{ex}(\mathcal{V})$ reduces to $\{W^\varkappa\} \cup \{\widehat{W}^\varkappa\}$ and hence P can be written as a convex combination of the measures P^\varkappa and \widehat{P}^\varkappa (the laws of W^\varkappa and \widehat{W}^\varkappa), $\varkappa = 0, 1, 2, \dots$. By Corollary 18, there exists a sequence of numbers $n_1 < n_2 < \dots$ such that $D(\Pi_{n_i})/(n-1) \rightarrow \frac{1}{2}$ almost surely. On the other hand the same ratio goes to 0 or 1 under the distribution P^\varkappa or \widehat{P}^\varkappa , respectively. This leads to a contradiction, so the proof is complete.

7 Concluding remarks

7.1 Permutations with descent-set statistic

We were led to consider the Eulerian triangle \mathcal{E} in connection with a larger graded graph \mathcal{Z} of zigzag diagrams [6]. With edge multiplicities taken into account, both graphs have the same path spaces, but \mathcal{Z} has more vertices and much more rich branching. The boundary problem for \mathcal{Z} amounts to describing all random arrangements $\Pi = (\Pi_n)$ with the property that the distribution of each Π_n is uniform conditionally given the set of descent positions in Π_n . In [6] we established that the distribution of a random total order determined by such Π must be spreadable, that is invariant under increasing mappings $\mathbb{N} \rightarrow \mathbb{N}$. D-arrangements are the simplest of this kind, and the

extreme D-arrangements we described here are also extreme solutions to the boundary problem for \mathcal{Z} .

Analogous connection exists between Kingman's graph \mathcal{K} of partitions and the Stirling triangle \mathcal{S} of the first kind [7]. The relevant random objects are exchangeable partitions of \mathbb{N} and a smaller class of partitions which have the number of blocks as sufficient statistic. In that case the situation is more interesting than the one for \mathcal{Z} and \mathcal{E} : extremes solutions to the boundary problem for \mathcal{S} (the celebrated Ewens partition structures) are decomposable along the boundary of \mathcal{K} , with the mixing measure being the remarkable Poisson–Dirichlet distribution [22].

7.2 A problem of moments

Corollary 3 and (7) tell us that every solution $V \in \mathcal{V}$ satisfies

$$V_{n0} = \sum_{\theta \in \Theta} p(\theta) \prod_{i=0}^{n-1} \frac{1 + (2\theta - 1)i}{1 + i}$$

for some unique probability distribution p on the parameter set Θ . An inverse problem asks one to characterize all sequences (V_{n0}) with $V_{10} = 1$ which can be represented in this form. An answer is suggested by the argument in Lemma 15 which says that there is a linear operator $\nabla : (V_{n0}) \mapsto (V_{nk})$ which maps an arbitrary sequence to a solution of (3). So the necessary and sufficient condition for representability is that ∇ applied to (V_{n0}) produces a nonnegative array.

The analogous question for Pascal's triangle is the Hausdorff moment problem, with kernel θ^n , where θ ranges in $[0, 1]$. In this classical case the analogue of ∇ associates with each sequence the array of its iterated differences, whose positivity is Hausdorff's condition called total monotonicity.

7.3 Remarks on the uniform case

The Eulerian numbers are given by the formula

$$\langle n \rangle_k = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n, \quad (22)$$

which compared with Laplace's formula in [5, Section 1.9] shows that

$$\text{Prob}(k \leq Y_1 + \dots + Y_n < k + 1) = \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle / n!$$

for Y_1, Y_2, \dots independent random variables with uniform distribution on $[0, 1]$. The following explanation of this coincidence is borrowed from [21, p. 296]. For $x > 0$ let $x = \lfloor x \rfloor + \{x\}$ be the decomposition of x into integer and fractional parts, with $0 \leq \{x\} < 1$. Consider

$$S_j = Y_1 + \dots + Y_j, \quad X_j = \{S_j\},$$

then X_1, X_2, \dots are also independent, uniform on $[0, 1]$. Observe that $X_j > X_{j+1}$ each time $\lfloor S_j \rfloor < \lfloor S_{j+1} \rfloor = \lfloor S_j \rfloor + 1$ and recall the discussion preceding Lemma 17.

Improving upon Corollary 18, we see that the convergence $D(\Pi_n)/(n - 1) \rightarrow \frac{1}{2}$ holds with probability 1. The central limit theorem applied to S_n 's entails that the distribution of $D(\Pi_n)$ is asymptotically Gaussian. This connection with sums of random variables has been a starting point for many fine results on descents in uniform permutation. See [19] for recent development and references.

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